UNIVERSITY OF KONSTANZ
Department of Physics
Thiago Lucena, Dr. Andrey Moskalenko

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Quantum Optics
Winter semester 2017/2018 - Exercise sheet 7
Distributed: 10.12.2018, Discussion: 17.12.2018

Problem 1: Phase states of light.

a) Show that the (continuous) phase distribution $P_\alpha(\phi) = \left|{}_{\text{ph}}\langle\phi|\alpha\rangle_{\text{coh}}\right|^2$ for a coherent state $|\alpha\rangle_{\text{coh}}$ is given by:

$$P_\alpha(\phi) = \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{(e^{-i\phi}\alpha)^n}{\sqrt{n!}}\right|^2.$$

b) Using the following property of the number representation of the phase operator $\hat{\phi} = -i\ln(\widehat{e^{i\theta}})$

$$\langle n|\hat{\phi}|\alpha\rangle = -i\frac{\partial}{\partial n}\langle n|\alpha\rangle$$

and considering $|\alpha| \gg 1$, show that:

$$\langle\alpha|\hat{\phi}|\alpha\rangle = \theta - \frac{i}{2}(\ln\langle n\rangle - \langle\ln(n)\rangle).$$

For $|\alpha| \gg 1$, the Poissonian distribution tends to a vanishing relative uncertainty: $P_\alpha(n) \to \delta_{n,\langle n\rangle}$. Use this limit to justify that $\langle\hat{\phi}\rangle \approx \theta$. HINT: use the asymptotic expansion of the harmonic number for $n \to \infty$, $H_n = \sum_{k=1}^{n}\frac{1}{k} = \ln(n) + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty}\frac{B_{2k}}{2kn^{2k}}$, where $\gamma \approx 0.58$ is the Euler-Mascheroni constant and the B_l are the Bernoulli numbers.

c) Using the same assumptions from the previous question, show that:

$$\langle\alpha|\hat{\phi}^2|\alpha\rangle \approx \theta^2 + \frac{1}{4\langle n\rangle}.$$

What is the product of the uncertainties $\Delta\phi\Delta n$?

Problem 2: Wigner function.

a) Using the expression for the Wigner function in terms of the density operator, $W(x,p) = \frac{1}{2\pi}\int dx'\langle x - \frac{x'}{2}|\hat{\rho}|x + \frac{x'}{2}\rangle e^{ipx'}$, calculate the Wigner function $W_\alpha(x,p)$ for a coherent state with density matrix $\hat{\rho} = |\alpha\rangle\langle\alpha|$.

b) Calculate the Wigner function $W_\beta(x,p)$ for a thermal state. HINT: use the completeness relations for Hermite polynomials (Mehler's formula).

c) Compare the results of $W_\alpha(x,p)$ and $W_\beta(x,p)$ for $\alpha = 0$ and $\langle n\rangle = 0$. What happens when $\langle n\rangle$ grows?